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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

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If m is a negative integer, then

$$1^{2m+1}\frac{\cosh a\pi}{\cosh 2a\pi}+3^{2m+1}\frac{\cosh 3a\pi}{\cosh 6a\pi}-- \dots \dots$$

$$+\frac{(-1)^{m+1}}{(4a)^{2m+2}}\left\{\frac{\cosh\left(\frac{\pi}{16a}\right)}{\cosh\left(\frac{\pi}{8a}\right)}+3^{2m+1}\frac{\cosh\left(\frac{3\pi}{16a}\right)}{\cosh\left(\frac{3\pi}{8a}\right)}-- \dots \dots\right\}$$

$$=\sqrt{2}\sum_r \frac{(4a)^{-r}\,L_2(1-r)\,L_2(r-2m-1)}{\frac{\pi}{2}\cos\frac{r\pi}{2}} \quad \dots \quad (1.7)$$

where r is given the values $0, -2, -4, -6, \dots + (2m+2)$.

The latter sum contains $(-m)$ non-vanishing terms.

§ **2.** Let m be equal to zero or a positive integer, and 'a' any real number, then by considering the integral

$$\int \frac{z^{4m+3}\,dz}{\cos z\,(1-i)\cos az\,(1+i)}$$

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along the following contour:

the x-axis from $(0, 0)$ to $(R, 0)$; the quadrant of a circle, centre the origin, joining $(R, 0)$ to $(0, R)$; the y-axis from $(0, R)$ to $(0, 0)$ where $R \to \infty$; we easily obtain the result:

$$\int_0^\infty x^{4m+3}\frac{(\cos x\cosh x\sin ax\sinh ax-\sin x\sinh x\cos ax\cosh ax)\,dx}{(\cos 2x+\cosh 2x)(\cos 2ax+\cosh 2ax)}$$

$$=-\frac{\pi}{8}\cdot\left(\frac{\pi}{2}\right)^{4m+3}(-4)^{m+1}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(2r-1)^{4m+3}}{\cosh(2r-1)\,\pi a/2}. \quad (2.1)$$

In particular*, $a = 1$ gives Ramanujan's result:

$$\frac{1^{4m+3}}{\cosh\frac{\pi}{2}}-\frac{3^{4m+3}}{\cosh\frac{3\pi}{2}}+\frac{5^{4m+3}}{\cosh\frac{5\pi}{2}}-+\dots=0.$$

---

* See also A and B.

By considering the integral

$$\int \frac{z^m e^{inz}\, dz}{\cosh z - \cos z}$$

where m is a positive integer greater than z, and n is a positive real number over the same contour, we easily obtain the result :

$$\int_0^\infty \frac{x^{4k+3} \sin nx\, dx}{\cosh x - \cos x} = \pi^{4k+4} (-4)^{k+1} \sum_{r=1}^{\infty} \frac{(-1)^r r^{4k+3} e^{-n\pi r} \sin n\pi r}{\sinh r\pi} \quad \ldots \quad (2.2)$$

where k is a positive integer or 0.

Hence if n is any positive integer*,

$$\int_0^\infty \frac{x^{4k+3} \sin nx\, dx}{\cosh x - \cos x} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

Now it is a well-known result in Cauchy's Theory of Reciprocal Functions (Cauchy : *Oeuvres*. Series 2, Vol. 7, pp. 176, *et seq.*) that if

$$\int_0^\infty \phi(x) \sin nx\, dx = \psi(n)$$

then

$$\frac{\pi}{2a} \left\{ \phi\left(\frac{\pi}{2a}\right) - \phi\left(\frac{3\pi}{2a}\right) + \phi\left(\frac{5\pi}{2a}\right) - + \ldots \right\}$$

$$= \psi(a) - \psi(3a) + \psi(5a) - \psi(7a) + - + - \ldots$$

where a is a real positive number.

From (2.3) and this result it follows that :

$$\frac{1^{4k+3}}{\cosh \frac{\pi}{2n} - \cos \frac{\pi}{2n}} - \frac{3^{4k+3}}{\cosh \frac{3\pi}{2n} - \cos \frac{3\pi}{2n}} + - \ldots = 0 \qquad (2.4)$$

where 'n' is any integer and 'k' is zero or a positive integer.

Now let $\phi(x)$ be a function of x such that

$$\int_0^\infty \phi(x) \sin(nx)\, dx = 0 \quad \ldots \quad \ldots \quad (2.5)$$

* Cf. Q. 353 of S. Ramanujan, in the *Jour. Indian Math. Soc.* or the obituary notice by G. H. Hardy, *Proc. Lond. Math. Soc.* (1922).

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for all integral values of n. Then we have

$$\int_0^\infty \phi(x) \sin(mx) \cos(nx)\, dx = 0$$

for all integral values of m and n, and in particular

$$\int_0^\infty \phi(x) \sin(bcx) \cos(nx)\, dx = 0 \qquad \ldots \quad (2.6)$$

for all integral values of b, c and n.

It is a well-known result in Cauchy's Theory of Reciprocal Functions, that if

$$\int_0^\infty f(x) \cos(nx)\, dx = g(n) \quad \ldots \qquad \ldots \quad (2.7)$$

then

$$\tfrac{1}{2} g(0) + \sum_{n=1}^{\infty} g(n\theta) = \frac{\pi}{\theta}\left[\tfrac{1}{2} f(0) + \sum_{n=1}^{\infty} f(2n\pi/\theta)\right] \quad (2.7)$$

Now let p be a prime number. In (2.7) put

$$f(x) = \phi(x) \sin(bcx);\quad \theta = pc.$$

In virtue of (2.6) we get:

$$\sum_{n=1}^{n=\infty} \sin\left(\frac{2nb\pi}{p}\right) \phi\left(\frac{2n\pi}{cp}\right) = 0 \ \ldots \qquad \ldots \quad (2.8)$$

where $\phi(x)$ is a function satisfying (2.5), b and c are any integers, while p is a prime.

Let

$$\begin{aligned} S_k = {} & \phi\left[\frac{2k\pi}{pc}\right] - \phi\left[\frac{2(p-k)\pi}{pc}\right] \\ & + \phi\left[\frac{2(p+k)\pi}{pc}\right] - \phi\left[\frac{2(2p-k)\pi}{pc}\right] \\ & \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ & + \phi\left[\frac{2(np+k)\pi}{pc}\right] - \phi\left[\frac{2(np+p-k)\pi}{pc}\right] \\ & + \ldots \qquad \ldots \qquad \ldots \qquad \ldots \end{aligned}$$

If we put $b = 1, 2, 3, \ldots \ldots \frac{1}{2}(p-1)$ in (2.8) we get:

$$\left.\begin{aligned}
&\sin\left(\frac{2\pi}{p}\right).S_1 + \sin\left(\frac{4\pi}{p}\right)S_2 + \ldots + \sin\left(\frac{2m\pi}{p}\right).S_m = 0\\
&\sin\left(\frac{4\pi}{p}\right).S_1 + \sin\left(\frac{8\pi}{p}\right)S_2 + \ldots + \sin\left(\frac{4m\pi}{p}\right).S_m = 0\\
&\quad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\\
&\sin\left(\frac{2m\pi}{p}\right).S_1 + \sin\left(\frac{4m\pi}{p}\right)S_2 + \ldots + \sin\left(\frac{2m^2\pi}{p}\right)S_m = 0
\end{aligned}\right\} \quad (2.9)$$

where $m = \frac{1}{2}(p-1)$.

From these m equations it will follow that

$$S_1 = S_2 = S_3 = \ldots = S_m = 0$$

if we can prove that the determinant :—

$$\begin{vmatrix}
\sin(\phi), & \sin(2\phi), & \ldots & \sin(m\phi)\\
\sin(2\phi), & \sin(4\phi), & \ldots & \sin(2m\phi)\\
\cdot & \cdot & \ldots & \cdot\\
\sin(m\phi), & \sin(2m\phi) & \ldots & \sin(m^2\phi)
\end{vmatrix} \ldots \quad (2.10)$$

has a value different from zero [where $\phi = \frac{2\pi}{p}$ and m is the same as in (2 9)]

Let us denote the absolute value of this determinant by Δ. When p is a prime $\equiv 3 \pmod{4}$, it is easy to see that $2^m . \Delta$ is equal to the circulant :—

$$\begin{vmatrix}
a_1 & a_2 & \cdot & \cdot & a_m\\
a_2 & a_3 & \cdot & \cdot & a_1\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
a_m & a_1 & \cdot & \cdot & a_{m-1}
\end{vmatrix}$$

where $m = \frac{1}{2}(p-1)$ and

$$a_k = r^{g^{2k}} - r^{-g^{2k}}$$

where g is a primitive root of the prime p, and r is any root, except unity, of the equation

$$x^p = 1.$$

The above determinant is equal to

$$\prod_{r=1}^{r=m} \{ a_1 + \theta^r a_2 + \theta^{2r} a_3 + \dots + a_m \theta^{(m-1)r} \} \quad \dots \quad (2.8)$$

where θ is a primitive root of the equation

$$x^m = 1.$$

Let us find the absolute value of

$$a_1 + \theta^s a_2 + \theta^{2s} a_3 + \dots + \theta^{(m-1)s} a_m$$

$$= \{ r^{g^2} + \theta^s r^{g^4} + \theta^{2s} r^{g^6} + \dots + \theta^{(m-1)s} r^{g^{p-1}} \}$$

$$- \{ r^{-g^2} + \theta^s r^{-g^4} + \theta^{2s} r^{-g^6} + \dots + \theta^{(m-1)s} r^{-g^{p-1}} \}.$$

Call this expression $f(r, \theta)$. Then the conjugate expression $f(r^{-1}, \theta^{-1})$ has the same absolute value.

Multiplying together the expressions for $f(r, \theta)$ and $f(r^{-1}, \theta^{-1})$ we get

$$| f(r, \theta) |^2 = 2m + b_1 r + b_2 r^2 + \dots + b_{2m} r^{2m} \qquad (2.11)$$

where the b's are independent of r.

Again, we have,

$$| f(r, \theta) | = | f(r^g, \theta) | = | f(r^{g^2}, \theta) | = \dots = | f(r^{g^{p-2}}, \theta) |$$

Hence,

$$| f(1, \theta) |^2 + | f(r, \theta) |^2 + | f(r^g, \theta) |^2 + \dots + | f(r^{g^{p-2}}, \theta) |^2$$

$$= (p-1) . | f(r, \theta) |^2 \quad \dots \quad \dots \quad (2.12)$$

since $f(1, \theta) = 0$.

If we make use of (2.11), the last result becomes

$$(p-1) . | f(r, \theta) |^2 = 2m . p \quad \dots \quad (2.13)$$

that is

$$| f(r, \theta) |^2 = p. \quad \dots \quad \dots \quad (2.14)$$

From (2.8) and (2.14) we find that

$$2^m \Delta = p^{\frac{1}{2}m} = p^{\frac{1}{2}(p-1)}$$

$$\Delta = 2^{\frac{1}{2}(1-p)} . p^{\frac{1}{2}(p+1)} \quad \dots \quad \dots \quad {}^{*}(2.15)$$

Hence

$$S_1 = S_2 = \dots = S_{\frac{1}{2}(p-1)} = 0 \quad \dots \quad (2.16)$$

when p is a prime $\equiv 3 \pmod{4}$.

From (2.3) we see that if k is a positive integer, the function

$$\phi(x) = \frac{x^{4k-1}}{\cosh x - \cos x} \quad \dots \quad \dots \quad (2.17)$$

satisfies the condition (2.5).

Take $p = 7$ and $k = 1$ in (2.16) and (2.17). We obtain :

$$\frac{1^3}{\cosh \frac{2\pi}{7n} - \cos \frac{2\pi}{7n}} - \frac{6^3}{\cosh \frac{12\pi}{7n} - \cos \frac{12\pi}{7n}} + \frac{8^3}{\cosh \frac{16\pi}{7n} - \cos \frac{16\pi}{7n}}$$

$$- \frac{13^3}{\cosh \frac{26\pi}{7n} - \cos \frac{26\pi}{7n}} + \frac{15^3}{\cosh \frac{30\pi}{7n} - \cos \frac{30\pi}{7n}} - \frac{20^3}{\cosh \frac{40\pi}{7n} - \cos \frac{40\pi}{7n}}$$

$$+ - + - \dots = 0. \quad (2.18)$$

(where the numbers 1, 8, 15, ... and 6, 13, 20, ... form two arithmetical progressions) which is the series for S_1. Combining the series for $+ S_1$, $+ S_2$ and $- S_3$ we get :

$$\sum_{n=1}^{n=\infty} \left(\frac{n}{7}\right) \cdot \frac{n^3}{\left\{ \cosh \frac{2n\pi}{7a} - \cos \frac{2n\pi}{7a} \right\}} = 0 \quad \dots \quad (2.19)$$

where $\left(\frac{n}{7}\right)$ is the Legendre symbol of quadratic residue modulo 7, and 'a' is any positive integer.

---

* A more general result has been stated by M. Bhimasena Rao and M. Venkatarama Aiyar. Vide *Jour. Ind. Math. Soc.*, Question **1294.**

Another result obtained by the same methods is :

$$\frac{1^{4k-3}}{\cos\frac{\pi}{4n}+\cosh\frac{\pi}{4n}}+\frac{3^{4k-3}}{\cos\frac{3\pi}{4n}+\cosh\frac{3\pi}{4n}}-\ldots\ldots=0 \qquad \ldots \quad (2.20)$$

where n is an even integer, but k is any integer.

Let us now integrate

$$\int\frac{z^{4k+3}}{(\cos z+\cosh z)\cos(az)}\,dz$$

(where k is any integer) over the same contour, as before.

We get :—

If k is any integer (zero, positive or negative), then

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(2r-1)^{4k+3}}{\left[\cos(2r-1)\frac{\pi}{2a}+\cosh(2r-1)\frac{\pi}{2a}\right]}+2.(-4)^{k+1}.a^{4k+4}\times$$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(2r-1)^{4k+3}\cos(2r-1)\frac{\pi a}{2}\cosh(2r-1)\frac{\pi a}{2}}{[\cos(2r-1)\pi a+\cosh(2r-1)\pi a]}$$

$$=\frac{1}{2}C_{-4k-4,\,a}\times\left(\frac{2}{\pi}\right)^{4k+3}a^{4k+4} \qquad \ldots \quad (2.21)$$

where $C_{n,a}$ is defined by the expansion

$$\frac{\sec ax}{\cos x+\cosh x}=\sum_{n=-\infty}^{n=+\infty}C_{n,a}\,x^{n}$$

so that $\qquad C_{0,a}=\frac{1}{2};\ C_{-n,a}=C_{2n-1,a}=0$

for all non-zero positive integral values of n.

As particular cases :

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(2r-1)^{4k+3}}{\cos(2r-1)\frac{\pi}{2a}+\cosh(2r-1)\frac{\pi}{2a}}=0 \qquad \ldots \quad (2.22)$$

if a is an odd integer, and $k=0, 1, 2, 3 \ldots$

and if $k = -1$, then

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)} \frac{1}{\left\{\cos(2r-1)\frac{\pi}{2a} + \cosh(2r-1)\frac{\pi}{2a}\right\}} = \frac{\pi}{8}$$

if a is an odd integer, a result first given by Ramanujan.

If a is an odd integer, then

$$\frac{1}{\cos\frac{\pi}{2a} + \cosh\frac{\pi}{2a}} - \frac{1}{3^5}\frac{1}{\cos\frac{3\pi}{2a} + \cosh\frac{3\pi}{2a}}$$

$$+ \frac{1}{5^5}\frac{1}{\cos\frac{5\pi}{2a} + \cosh\frac{5\pi}{2a}} - + \ldots = \frac{(5a^4 - 1)\,\pi^5}{768 \times 4} \quad \ldots \quad (2.23)$$

The particular case $a = 1$ has been given by Ramanujan.*

Again if r is a positive real number

$$\int_0^\infty \frac{(1 - \cos 2rx)\sin x\,dx}{x(\cosh x - \cos x)}$$

$$= 2\sum_{n=1}^{n=\infty} \left\{ \int_0^\infty \frac{e^{-nx}\sin nx}{x}(1 - \cos 2rx)\,dx \right\}$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2r^2}{n^2}\right).$$

It follows that

$$2\int_0^\infty \frac{\sin^2 rx \sin x\,dx}{x(\cosh x - \cos x)} = r\pi - \frac{\pi}{4}$$

if r is a positive integer.

Change r into $r + 1$ in this result, and subtract.

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* As cited in the obituary notice by G H. Hardy. *Pro Lond Math. Soc*. 1922,

We get

$$\int_0^\infty \frac{\sin^2 x}{x} \frac{\sin (2r-1) x \, dx}{(\cosh x - \cos x)} = \frac{\pi}{2} \qquad \ldots \quad (2.24)$$

if r is a positive integer greater than 1.

If $r = 1$,

$$\int_0^\infty \frac{\sin^3 x}{x} \frac{dx}{(\cosh x - \cos x)} = \frac{3\pi}{8}.$$

Applying Cauchy's Theory of Reciprocal Functions, we get:

$$\frac{\sin^2 \frac{\pi}{2n}}{\left(\cosh \frac{\pi}{2n} - \cos \frac{\pi}{2n}\right)} - \frac{1}{3} \frac{\sin^2 \frac{3\pi}{2n}}{\left(\cosh \frac{3\pi}{2n} - \cos \frac{3\pi}{2n}\right)} + - \ldots = \frac{\pi}{4} \quad (2.25)$$

if n is an odd integer greater than unity.

If $n = 1$ we get a result which is contained in the second of the formulae (2.9).

The following are a few other formulae, obtained by the methods of this section:

If n is an odd positive integer, then

$$\frac{1}{\left(\cosh \frac{\pi}{4n} + \cos \frac{\pi}{4n}\right)} + \frac{1}{3} \frac{1}{\left(\cosh \frac{3\pi}{4n} + \cos \frac{3\pi}{4n}\right)} - \ldots = \frac{\pi}{4\sqrt{2}}. \quad (2.26)$$

$$\frac{\sin^{2k} \left(\frac{\pi}{2n}\right)}{\left(\cos \frac{\pi}{2n} + \cosh \frac{\pi}{2n}\right)} - \frac{1}{3} \frac{\sin^{2k} \left(\frac{3\pi}{2n}\right)}{\left(\cos \frac{3\pi}{2n} + \cosh \frac{3\pi}{2n}\right)} + - \ldots = 0 \quad (2.27)$$

if n is an odd positive integer greater than $2k$, and k is a positive non-zero integer.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2n-1)^{4k+1}}{\cos (2n-1) \frac{\pi}{2a} + \cosh (2n-1) \frac{\pi}{2a}} = 0 \quad \ldots \quad (2.28)$$

if a is an even positive integer and k is zero or a positive integer.

§ **3.** By contour integration it is easy to prove that

$$\int_0^\infty \frac{x\,dx}{\cos x + \cosh x} = \frac{\pi^2}{2} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(2r-1)}{\cosh (2r-1)\frac{\pi}{2}};$$

$$\int_0^\infty \frac{x^5\,dx}{\cos x + \cosh x} = -\frac{\pi^6}{8} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(2r-1)^5}{\cosh (2r-1)\frac{\pi}{2}};$$

and so on.

Again with the usual notation in elliptic functions:

$$k\left(\frac{2k}{\pi}\right)^2 \operatorname{sn}\left(\frac{2kx}{\pi}\right) \operatorname{dn}\left(\frac{2kx}{\pi}\right) = 4\sum_{r=1}^{\infty} \frac{(2r-1)\,q^{\frac{2r-1}{2}}}{1+q^{2r-1}} \sin (2r-1)\,x$$

whence

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}(2r-1)^5}{\cosh (2r-1)\frac{\pi}{2}} = \left\{-\frac{k}{2}\left(\frac{2k}{\pi}\right)^6 \frac{d^5}{dx^5}(\operatorname{cn} x)\right\}_{x=K,\ k=\frac{1}{\sqrt{2}}}$$

But if $k = \frac{1}{\sqrt{2}}$, $K = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$.

Hence we obtain results like the following:

$$\int_0^\infty \frac{x\,dx}{\cos x + \cosh x} = \frac{1}{128\pi}\{\Gamma(\tfrac{1}{4})\}^4 \quad \ldots \quad (3.1)$$

$$\int_0^\infty \frac{x^5}{\cos x + \cosh x}\,dx = \frac{3}{2048\pi^3}\{\Gamma(\tfrac{1}{4})\}^{12} \quad \ldots \quad (3.2)$$

Again, when k is a positive integer, we integrate the function

$$\frac{z^{4k-1}}{\cosh z - \cos z}$$

over the contour of § 2, and obtain

$$\int_0^\infty \frac{x^{4k-1}}{\cosh x - \cos x}\,dx = \tfrac{1}{2}\,\pi^{4k}.(-4)^k \sum_{n=1}^{n=\infty} \frac{(-1)^n\,n^{4k-1}}{\sinh (n\pi)} \quad \ldots \quad (3.3)$$

Now, we know that,

$$\left(\frac{2kK}{\pi}\right)^2 \text{cn}^2\left(\frac{2Kx}{\pi}\right) = 8\sum_{n=1}^{n=\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cdot \cos(2nx).$$

From this it is easy to deduce that

$$\sum_{1}^{\infty} \frac{(-1)^{n-1}\, n^3}{\sinh(n\pi)} = \frac{1}{512\pi^6}\,\{\Gamma(\tfrac{1}{4})\}^8 \qquad \ldots \quad (3.4)$$

From this and (3.3) we get

$$\int_0^\infty \frac{x^3}{\cosh x - \cos x}\,dx = \frac{1}{256\pi^2}\,\{\Gamma(\tfrac{1}{4})\}^8 \qquad \ldots \quad (3.5)$$

§ **4.** In this section, some interesting congruence properties of the co-efficients in certain elliptic function expansions, are noted.

We know that (see A)

$$1^m \operatorname{sech}(\tfrac{1}{2}\pi) - 3^m \operatorname{sech}(\tfrac{3}{2}\pi) + 5^m \operatorname{sech}(\tfrac{5}{2}\pi) - + \ldots = 0. \quad (4.1)$$

when $m \equiv 3$ (mod. 4) is positive, and

$$1^m \operatorname{sech}\left(\frac{\sqrt{3}}{2}\pi\right) - 3^m \operatorname{sech}\left(\frac{3\sqrt{3}}{2}\pi\right) + 5^m \operatorname{sech}\left(\frac{5\sqrt{3}}{2}\pi\right) - + \ldots = 0 \qquad \ldots \quad (4.2)$$

when $m \equiv 5$ (mod 6) is positive.

Again

$$\frac{2kK}{\pi}\,\text{cn}\left(\frac{2Kx}{\pi}\right) = 2\sum_{n=1}^{n=\infty} \operatorname{sech}\left\{(2n-1)\frac{\pi K'}{2K}\right\} \cos(2n-1)\,x. \quad (4.3)$$

From (4.1) and (4.3) we have

$$\left[D^m\left(\text{cn}\,\frac{2Kx}{\pi}\right)\right]_{x=\pi/2,\; k=1/\sqrt{2}} = 0 \qquad \ldots \quad (4.4)$$

where $m \equiv 3$ (mod. 4) is positive, and $D \equiv \dfrac{d}{dx}$

In other words:

$$[D^m(\text{cn}\,x)]_{x=K} = k'\,F(k) \qquad \ldots \quad (4.5)$$

where F (k) is a polynomial in k which is divisible by $k - \frac{1}{\sqrt{2}}$ and hence also by $(2k^2 - 1)$, since the co-efficients in F (k) are integers $[m \equiv 3 \pmod{4}]$.

Similarly, (4.2) and (4.3) give the following :—

$$[D^m (\text{cn } x)]_{x=K} = k' G(k)$$

where $G(k)$ is a polynomial in k which has $(k - \sin 15^\circ)$ or $(k + \frac{1}{2} - \frac{1}{2}\sqrt{3})$ as a factor. Similarly, $G(k)$ has the factor $(k + \sin 15^\circ)$. Hence it has the factor

$$(k^2 - \sin^2 15) = \{ k^2 - \tfrac{1}{8}(4 - 2\sqrt[2]{3}) \}.$$

From the last fact it follows that G (k) is also divisible by $\{ k^2 - \frac{1}{8} \{ 4 + 2\sqrt{3} \}$ and hence by the product of these two factors, that is, by

$$k^4 - k^2 + \frac{1}{16}. \; [m \equiv 5 \pmod{6}.]$$

From (4.5) and (4.6) it follows that :—

If $m \equiv 11 \pmod{12}$, and if

$$[D^m (\text{cn } x)]_{x=K} = k' F(k) \quad \ldots \quad \ldots \quad (4.7)$$

then F (k) has the factor $(16k^4 - 16k^2 + 1)(2k^2 - 1)$.

This is equivalent to :—

$$\text{If} \quad \frac{\text{sn } (x, k)}{\text{dn } (x, k)} = \sum_{n=1}^{n=\infty} a_{n,k} \, x^n \quad \ldots \quad \ldots \quad (4.8)$$

then $\qquad a_{11,k}, a_{23,k}, a_{35,k}, \ldots a_{12m+11,k} \ldots$

are polynomials in k, and are divisible by

$$32k^6 - 48k^4 + 18k^2 - 1.$$

Incidentally, we may note the following fact.

In (4.6) we found that :

$$0 = [D^m (\text{cn } x)]_{x=K,\, k=\sin 2\pi/5} \quad \ldots \quad (4.9)$$

This combined with (4.3) leads to the following result:

$$1^m \operatorname{sech}\left(\frac{\pi}{2\sqrt{3}}\right) - 3^m \operatorname{sech}\left(\frac{3\pi}{2\sqrt{3}}\right) + 5^m \operatorname{sech}\left(\frac{5\pi}{2\sqrt{3}}\right) - + \dots = 0 \quad (4.10)$$

when $m \equiv 5 \pmod{6}$ is positive.

§ 5. Let $d(n)$ denote the number of divisors of n. It follows from Herr Mellin's Integral, that

$$\sum_{1}^{\infty} \frac{d(n)}{e^{ny} - 1} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \{ \Gamma(s) y^{-s} \zeta^3(s) \} \, ds$$

where $k > 1$, $R(y) > 0$.

Now

$$\Gamma(s) = \Gamma(1) - (s-1)\gamma + \frac{(s-1)^2}{2!}\left(\gamma^2 + \frac{\pi^2}{6}\right)$$

$$- \frac{(s-1)^3}{3!}\{\gamma^3 + 3\gamma\,\zeta(2) + 2\zeta(3)\} + \dots$$

$$y^{-s} = \frac{1}{y}\left[1 - (s-1)\log y + \frac{(s-1)^2}{2}\log^2 y - + \dots\right]$$

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2(s-1)^2 - \gamma_3(s-1)^3 + - + \dots$$

where γ is Euler's constant, and

$$r!\,\gamma_r = \lim_{n\to\infty}\left\{\frac{\log^r 1}{1} + \frac{\log^r 2}{2} + \dots + \frac{\log^r(n)}{n} - \frac{\log^{r+1}(n)}{r+1}\right\}.$$

Now $\dfrac{1}{2\pi i}\displaystyle\int_{k-i\infty}^{k+i\infty} \Gamma(s)\, y^{-s} \zeta^3(s)\, ds$

$$= -\frac{1}{2\pi i}\int_{-2m-3+\varepsilon-i\infty}^{-2m-3+\varepsilon+i\infty} \{\Gamma(s)\, y^{-s} \zeta^3(s)\}\, ds$$

+ { the sum of the residues of the function $\Gamma(s)\, y^{-s} \zeta^3(s)$ at all its poles between the lines $\sigma = k$ and $\sigma = -2m - 3 + \varepsilon$ }

where m is a *large* positive integer, ε is an arbitrarily small real number.

Now if y is a real number less than 1, the absolute value of the integral on the right-hand side of the last equation is less than

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} y^{2m+3-\varepsilon}\,|\,\Gamma(s)\,|\;\;|\,\zeta^3(s)\,|\;dt = O\left(y^{2m+3-\varepsilon}\right)$$

where the letter O stands for the 'order' symbol (Landau: *Handbuch.*)

From the five equations written down, we at once deduce the result :

$$\sum_1^\infty \frac{d(n)}{e^{ny}-1} = \frac{1}{y}\left[\frac{\log^2 y}{2} - 2\gamma\log y + \frac{\gamma^2}{2} - 3\gamma_1 + \frac{\pi^2}{12}\right]$$

$$-\frac{1}{8} + \sum_{r=0}^{r=m}\left[y^{2r+1}\left\{\frac{B_{r+1}}{2(r+1)}\right\}^3 \frac{(-1)^r}{2r+1}\right] + O\left(y^{2m+3-\varepsilon}\right)$$

(for all $\varepsilon > 0$) where B_r is the 'rth' Bernoullian Number.

Again, if $\sigma > 1$, it is known that *

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_1^\infty \frac{d^2(n)}{n^s}$$

From this and Mellin's Integral, it follows that

$$\sum_1^\infty \frac{d^2(n)}{e^{ny}-1} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\left\{\frac{\Gamma(s)\,\zeta^5(s)}{y^s\,\zeta(2s)}\right\}ds$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left\{\frac{\Gamma(s)\,\zeta^5(s)}{y^s\,\zeta(2s)}\right\}ds$$

$$+ \text{ residue of } \frac{\Gamma(s)\,\zeta^5(s)}{y\;\zeta(2s)} \text{ at the pole } s = 1.$$

Making use of the equations in the beginning of § 5, and also the result,

$$\frac{1}{\zeta(2s)} = \frac{1}{\zeta(2)} - \frac{2(s-1)\,\zeta'(2)}{\zeta^2(2)}$$

$$+\; 2(s-1)^2\left\{\frac{2\{\zeta'(2)\}^2}{\zeta^3(2)} - \frac{\zeta''(2)}{\zeta^2(2)}\right\} + \ldots$$

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* S. Ramanujan : *Mess. of Maths.* V. 45, pp. 81—84 (81).
B. M. Wilson : *Proc. Lond. Math. Soc.*, V. 21, 1923, pp. 235, *et seq.*

we find that the co-efficient of $\frac{1}{s-1}$ in the expansion (valid in the neighbourhood of $s = 1$) of

$$\frac{\Gamma(s)\,\zeta^5(s)}{y^s\,\zeta(2s)},$$

is
$$(a\log^4 y + b\log^3 y + c\log^2 y + d\log y + e)/y$$

where a, b, c, d, e are known, but complicated constants.

Now since

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin s\pi}$$

it follows that

$$|\Gamma(\tfrac{1}{2}+it)| \sim \frac{\sqrt{2\pi}}{e^{\frac{1}{2}|t|}}$$

when $|t| \to \infty$.

Again when $\sigma > -1$,

$$\zeta(s) = 1 + \frac{1}{s-1} - \frac{s}{2}\{\zeta(s+1)-1\} - \frac{s(s+1)}{2!}\sum_{1}^{\infty}\int_0^1 \frac{u^2\,du}{(n+u)^{s+2}}$$

so that

$$|\zeta(\tfrac{1}{2}+it)| = O(t^2) = |\zeta(\tfrac{1}{2}-it)|$$

Also it is known* that when $|t|$ is large

$$\left|\frac{1}{\zeta(1+it)}\right| = O\{\log^7(|t|)\}$$

It follows that

$$\frac{\Gamma(s)\,\zeta^5(s)}{y^s\,\zeta(2s)} = y^{-\frac{1}{2}}\,O\left(\frac{|t|^{10}\log^7|t|}{e^{\frac{|t|}{2}}}\right)$$

when $\sigma = \frac{1}{2}$; and hence that

$$\left|\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left\{\frac{\Gamma(s)\,\zeta^5(s)}{y^s\,\zeta(2s)}\right\}ds\right| = O\left(\frac{1}{\sqrt{y}}\right)$$

---

* Landau: *Handbuch*, Vol. I, p. 178.

Thus we have arrived at the result :

$$\sum_{1}^{\infty} \frac{d^2(n)}{e^{ny}-1} = \left\{ \frac{a \log^4 y + b \log^3 y + c \log^2 y + d \log y + e}{y} \right\} + O\left(\frac{1}{\sqrt{y}}\right) \qquad \dots \quad (5.2)$$

where a, b, c, d, e are known constants.

In general, if $\sigma > 1$, and 2^r is an integer,

$$\sum_{n=1}^{n=\infty} \frac{d^r(n)}{n^s} = \prod_p \left(1 + \frac{2^r}{p^s} + \frac{3^r}{p^{2s}} + \frac{4^r}{p^{3s}} + \dots\right)$$

$$= \{\zeta(s)\}^{2r} \phi(s)$$

where $\phi(s)$ is absolutely convergent of $\sigma > \frac{1}{2}$.

Consequently, if r is a positive integer, we have the following results :

$$\sum_{u=1}^{n=\infty} \frac{d^r(n)}{e^{ny}-1} = \left\{ \frac{a_0 (\log y)^{2^r} + a_1 (\log y)^{2^r-1} + a_{2^r}}{y} \right\} + O\left(\frac{1}{\sqrt{y}}\right) \qquad \dots \quad (5.3)$$

$$\sum_{n=1}^{\infty} d^r(n)\, (e^{-ny} = \left\{ \frac{b_0 (\log y)^{2^r-1} + b_1 (\log y)^{2^r-2} + \dots + b_{2^r-1}}{y} \right\}$$

$$+ O\left(\frac{1}{\sqrt{y}}\right) \qquad \dots \quad (5.4)$$

where the a's and b's are known constants, and y is a small positive number.

In particular, as $x \to 1$, we have

$$\sum_{n=1}^{\infty} \frac{d^3(n)\, x^n}{1-x^n} \sim \frac{1}{8!} \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{4}{p^5} + \frac{1}{p^6}\right)$$

$$\times \frac{\left\{\log\left(\frac{1}{1-x}\right)\right\}^8}{1-x}$$

where p runs through all the primes 2, 3, 5, 7 ... ... Again if $\sigma_a(n)$ denotes the sum of the 'ath' powers of the divisors of n, it is known* that

$$\sum_1^\infty \frac{\sigma_a(n)\,\sigma_b(n)}{n^s} = \frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)}$$

$[s = \sigma + it;\ \sigma > 1;\ \sigma > 1 + R(a);\ \sigma > 1 + R(b);\ \sigma > 1 + R(a+b)]$

Hence from Herr Mellin's Integral:

$$\sum_1^\infty \frac{\sigma_a(n)\,\sigma_b(n)}{e^{ny}-1} = \frac{1}{2\pi i} \times$$

$$\int_{k-i\infty}^{k+i\infty} \left\{ \frac{\Gamma(s)\,\zeta^2(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)\,\zeta(2s-a-b)}{y^s\,\zeta(2s-a-b)} \right\} ds$$

where $k > 1;\ k > 1 + R(a);\ k > 1 + R(b):\ k > 1 + R(a+b)$

Proceeding as before we obtain results like the following:

$$\sum_1^\infty \frac{\sigma_{10}(n)\,\sigma_{15}(n)}{e^{ny}-1} = \frac{\Gamma(26)\,\zeta^2(26)\,\zeta(16)\,\zeta(11)}{y^{26}\,\zeta(27)}$$

$$+ \frac{\Gamma(16)\,\zeta^2(16)\,\zeta(6)\,\zeta(-9)}{\zeta(7)\,y^{16}} + O\left(\frac{1}{y^{13}}\right) \quad \ldots \quad (5.5)$$

when y approaches zero.

$$\sum_1^\infty \frac{\sigma_a(n)\,\sigma_b(n)\,x^n}{1-x^n} \sim \frac{\Gamma(a+b+1)\,\zeta^2(a+b+1)\,\zeta(a+1)\,\zeta(b+1)}{\zeta(a+b+2)\,(1-x)^{a+b+1}}$$

when $0 < a < b$, and $x \to 1$.

$$\sum_1^\infty \frac{\sigma_5(n)\,\sigma_7(n)}{e^{ny}-1} = \frac{\Gamma(13)\,\zeta(8)\,\zeta(6)\,\zeta^2(13)}{\zeta(14)\,y^{13}} + O\left(y^{-\frac{13}{2}}\right).$$

as $y \to 0$.

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* S. Ramanujan: *Mess of Maths*, Vol. **45**, pp. **81—84 (81)**.

B. M. Wilson: *Proc, Lond. Math. Soc.*, Vol. **21**, **1923**, pp. 235, et seq.

Again let $\omega(n)$ denote the number of distinct prime factors of n; it is easy to verify hatt

$$\sum_{n=1}^{n=\infty} \frac{r^{\omega(n)}}{n^s} = \zeta^r(s)\,\phi(s)$$

$$\sum_{n=1}^{\infty} \frac{d(n^r)}{n^s} = \zeta^{r+1}(s)\,\phi(s)$$

where r is a positive integer, and $\phi(s)$ is an absolutely convergent Dirichlet Series, when $\sigma > \frac{1}{2}$. These formulae enable us to write down the dominant terms in the asymptotic expansions of

$$\sum_{1}^{\infty} \frac{r^{\omega(n)}}{e^{ny}-1} \text{ and } \sum_{1}^{\infty} \frac{d(n^r)}{e^{ny}-1}$$

as $y \to 0$.

In particular, as $x \to 1$,

$$\sum_{1}^{\infty} \frac{r^{\omega(n)}\,x^n}{1-x^n} \sim \frac{B_r}{r!} \frac{\{\log(1/1-x)\}^r}{1-x} \qquad \ldots \quad (5.6)$$

$$\sum_{1}^{\infty} \frac{d(n^r)\,x^n}{1-x^n} \sim \frac{B_r}{(r+1)!} \frac{\{\log(1/1-x)\}^{r+1}}{1-x} \qquad \ldots \quad (5.7)$$

where

$$B_r = \prod_p \left\{ \left(1+\frac{r-1}{p}\right)\left(1-\frac{1}{p}\right)^{r-1} \right\}$$

and p runs through all primes (2, 3, 5 ...).

§ 6. Let y and k be real numbers such that $0 < y < 1, 0 < k < 1$. Consider an infinite circle (whose centre is at the point $\sigma = k$, $t = 0$, and diameter the line $\sigma = k$) extending over the second and third quadrants. By evaluating the residues of $\dfrac{y^{-s}}{\sin s\pi}$ at $s = 0, -1, -2, -3 \ldots$, we obtain:

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \left\{ \frac{ds}{y^s \sin s\pi} \right\} = \frac{1}{\pi(1+y)}$$

where the path of integration is the straight line $\sigma = k$.

It follows that if $k > 1/r$, $r > 1$, then

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{1+n^r y} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\left\{\frac{y^{-s}\zeta(rs)\,ds}{\sin s\pi}\right\}$$

Proceeding as before, we find that if $y \to 0$, then*

$$\sum_{1}^{\infty}\frac{1}{1+n^r y^r} = \frac{\pi \operatorname{cosec} \pi/r}{ry} - \frac{1}{2}$$

$$-2\pi\left[\frac{\Gamma(r+1)\,\zeta(r+1)\,y^r}{(2\pi)^{r+1}\sec(r+1)\,\pi/2} + \frac{\Gamma(2r+1)\,\zeta(2r+1)\,y^{2r}}{(2\pi)^{2r+1}\sec(2r+1)\,\pi/2} + \dots\right.$$

$$\left. + \frac{\Gamma(mr+1)\,\zeta(mr+1)\,y^{mr}}{(2\pi)^{mr+1}\sec(mr+1)\,\pi/2}\right] + O\left(y^{mr+r-\varepsilon}\right) \quad \dots \quad (6.1)$$

for all $\varepsilon > 0$.

$$\sum_{1}^{\infty}\frac{d(n)}{1+n^r y^r} = \pi\left\{\frac{\log(1/y^r)}{r^2 y} + \frac{2\gamma}{ry}\right\}$$

$$+\tfrac{1}{4} + 4\left[\sum_{k=1}^{k=m}\frac{(-1)^k y^{kr}\,\Gamma^2(kr+1)\,\zeta^2(kr+1)}{(2\pi)^{kr+1}\sec^2\{(kr+1)\,\pi/2\}}\right]$$

$$+ O\left(y^{mr+r-\varepsilon}\right) \quad \dots \quad (6.2)$$

and so on

The following are a few other results obtained by the methods of §§ 5 and 6.

Let $d_k(n)$ denote the number of decompositions of n into k factors, so that $d_2(n) = d(n)$ and

$$\sum_{1}^{\infty}\frac{d_k(n)}{n^s} = \{\zeta(s)\}^k \qquad [\sigma > 1] \quad \dots \quad (6.3)$$

---

* As a particular case of (6.1) it should be noted that if $n > 1$,

$$\int_0^{\infty}\frac{dx}{1+x^n} = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}.$$

then

$$\sum_{n=1}^{n=\infty} \frac{d_k(n)}{e^{ny}-1} = \left\{ \frac{a_0 \log^k(y) + a_1 \log^{k-1}(y) + \dots + a_k}{y} \right\}$$

$$+ (-\tfrac{1}{2})^{k+1} + \sum_{r=0}^{r=m} \left[y^{2r+1} \frac{(-1)^{rk+k+r}}{\Gamma(2r+2)} \left\{ \frac{B_{r+1}}{2(r+1)} \right\}^{k+1} \right]$$

$$+ O(y^{2m+3-\varepsilon})$$

where B_r is the 'r'th Bernoullian number, and the first term in brackets on the right-hand side, is the residue at the pole $s = 1$, of the function

$$\Gamma(s)\, y^{-s}\, \zeta^{k+1}(s).$$

The case $k = 1$ is well-known.

If $x \to 1$, then* if r is a positive integer

$$\sum_{n=1}^{n=\infty} \{ d(n)\, d(n^2)\, d(n^3) \dots d(n^r)\, x^n \} \sim A_0 \frac{\{\log(1/1-x)\}^{\Gamma(r+2)-1}}{1-x}$$

where A_0 is a positive non-zero constant depending on r. ... (6.4)

If $\psi(n)$ denotes the number of triangular numbers (1, 3, 6 ... ...) $\leqslant \frac{n(n+1)}{2}$, which are prime to n, then as $x \to 1$

$$\sum_{1}^{\infty} \frac{\psi(n)\, x^n}{1-x^n} \sim \pi \left(\frac{1-2/p^2}{1-1/p^2} \right) \cdot \frac{1}{(1-x)^2} \qquad \dots \quad (6\,5)$$

where p runs through all the odd primes.

---

* By a well-known theorem due to Cesaro, we can also obtain this result from the asymptotic formula for

$$S(n = \sum_{x \leqslant n} \{ d(x)\, d(x^2) \dots d(x^r) \}$$

Now from a classical formula for the sum of the co-efficients in a Dirichlet Series (Landau : *Handbuch*, p. 346) we find that the dominant term in S (n) is $A_0 . n\ (\log n)^{\Gamma(r+2)-1}$. This proves (6.4).

We can prove (6.5) similarly, but *not* (6.1) or (6.3). [For the theorem of Cesaro referred to, see Bromwich : *Inf. Series*, pp. 131—133 ond pp. 171-172].

I will conclude this paper with a transformation, of infinite series, drawn from the theory of the L-function (Landau: *Handbuch*, pp 391, *et seq.*)

Let p be a prime of the form $4k+3$.

Let $\left(\frac{n}{p}\right)$ denote the Legendre quadratic-residue symbol defined before, and write

$$\mathrm{L}(s)=\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\frac{1}{n^s}\qquad[\sigma>1],$$

and its continuations.

From Herr Mellin's Integral, it is clear that:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{n}{p}\right)}{n\,(e^{ny}+e^{-ny})}=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\{\Gamma(s)\,\eta\,(s)\mathrm{L}(s+1)\,y^{-s}\}\,ds\quad[k>1]$$

Now we know that

$$\Gamma(s)\,\eta\,(s)\sin\frac{s\pi}{2}=\left(\frac{\pi}{2}\right)^{s}\eta\,(1-s)\qquad\ldots\qquad\text{(i)}$$

$$\Gamma(s)\,\mathrm{L}\,(s)\sin\frac{s\pi}{2}=\sqrt{\frac{p}{4}}\left(\frac{\pi}{4}\right)^{s}\mathrm{L}\,(1-s)\qquad\ldots\qquad\text{(ii)}$$

Hence using the first equation, we get

$$\sum_{n=1}^{\infty}\frac{\left(\frac{n}{p}\right)}{n\,(e^{ny}+e^{-ny})}=\frac{1}{2\pi i}\int_{-k-i\infty}^{-k+i\infty}\left\{\frac{\eta\,(1+s)\,\mathrm{L}\,(1-s)\left(\frac{2y}{\pi}\right)^{s}}{-\sin\frac{s\pi}{2}}\right\}ds$$

$$\therefore\sqrt{\frac{p}{4}}\sum_{n=1}^{\infty}\frac{\left(\frac{n}{p}\right)}{n\left(e^{\frac{n\pi^2}{py}}+e^{\frac{-n\pi^2}{py}}\right)}$$

$$=\frac{1}{2\pi i}\int_{-k-i\infty}^{-k+i\infty}\left\{\frac{\sqrt{\frac{p}{4}}\,\eta\,(1+s)\,\mathrm{L}\,(1-s)\,(2\pi/py)^{s}}{-\sin\frac{s\pi}{2}}\right\}ds\;\ldots\qquad\text{(iii)}$$

Again it is easy to see that:

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \tan^{-1}(e^{-ny}) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \left\{ \Gamma(s)\, \eta(s+1)\, L(s)\, y^{-s}\, ds \right\}$$

$$= \frac{1}{2\pi i} \sqrt{\frac{p}{4}} \int_{k+i\infty}^{k+i\infty} \left\{ \frac{\eta(s+1) \left(\frac{2\pi}{py}\right)^{s} L(1-s)}{\sin \frac{s\pi}{2}} \right\} ds \quad \ldots \text{(iv)}$$

from (ii).

From (iii) and (iv); we get

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \tan^{-1}[e^{-ny}] + \frac{\sqrt{p}}{4} \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right)}{n \cosh [n\pi^2/py]}$$

$$= \frac{1}{2\pi i} \sqrt{\frac{p}{4}} \left\{ \int_{k-i\infty}^{k+i\infty} - \int_{-k-i\infty}^{-k+i\infty} \right\} \left\{ \frac{(2\pi/py)^s\, \eta(1+s)\, L(1-s)}{\sin s\pi/2} \right\} ds$$

The second member of this equation is equal to

$$\sqrt{\frac{p}{4}} \frac{\eta(1)\, L(1)}{\pi/2}$$

since the only pole of the integrand, between the lines $\sigma = \pm k$, is at $s = 0$

Now let

$$S(x) = \frac{\pi - x}{2} = \sum_{1}^{\infty} \frac{\sin nx}{n}$$

Then

$$\sum_{n=1}^{p-1} \left\{ S\left[\frac{2n\pi}{p}\right] \cdot \left(\frac{n}{p}\right) \right\} = \sqrt{p} \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right)}{n} = \sqrt{p}\, L(1)$$

from the theory of Gauss' sums.

Hence, we derive the known result,

$$L(1) = -\pi \left[\sum_{n=1}^{n=p-1} \left(\frac{n}{p}\right) \cdot n \right] \Big/ p\sqrt{p}.$$

Thus we have proved that:—

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \tan^{-1}\left[e^{-ny}\right] + \frac{\sqrt{p}}{4} \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right)}{n \cosh\left[n\pi^2/py\right]}$$

$$= -\frac{\pi}{4p}\left[\sum_{n=1}^{n=p-1} \left(\frac{n}{p}\right) + n\right]$$

Again, the L-function introduced in § 1, satisfies the functional equation:—

$$2\,\Gamma(s)\, L_2(s) \sin\left(\frac{s\pi}{2}\right) = \sqrt{8}\left(\frac{\pi}{4}\right)^{s} L_2(1-s).$$

which is exactly the same form as (ii) above, in case p is put equal to 8.

Thus an exactly similar reasoning shows that *:

$$\tan^{-1}[e^{-\pi y}] + \tan^{-1}[e^{-3\pi y}] - \tan^{-1}[e^{-5\pi y}] - \tan^{-1}[e^{-7\pi y}]$$
$$+ + - - \dots \dots$$
$$+ \frac{1}{\sqrt{2}}\left\{\operatorname{sech}\frac{\pi}{8y} + \frac{1}{3}\operatorname{sech}\frac{3\pi}{8y} - \frac{1}{5}\operatorname{sech}\frac{5\pi}{8y} - \frac{1}{7}\operatorname{sech}\frac{7\pi}{8y}\right.$$
$$\left. + + - - \dots \dots\right\} = \frac{\pi}{4} \quad \dots \quad (7.2)$$

As a particular case of (7.1) we have the result:

$$\tan^{-1}(e^{-\pi}) + \tan^{-1}(e^{-2\pi}) - \tan^{-1}(e^{-3\pi}) + \tan^{-1}(e^{-4\pi})$$
$$- \tan^{-1}(e^{-5\pi}) - \tan^{-1}(e^{-6\pi}) + \tan^{-1}(e^{-8\pi}) + \tan^{-1}(e^{-9\pi})$$
$$- \tan^{-1}(e^{-10\pi}) + \tan^{-1} e^{-11\pi}) - \tan^{-1}(e^{-12\pi}) - \tan^{-1}(e^{-13\pi})$$
$$+ + - + - \dots \dots$$
$$+ \frac{\sqrt{7}}{4}\left\{\operatorname{sech}\frac{\pi}{7} + \frac{\operatorname{sech}\left(\frac{2\pi}{7}\right)}{2} - \frac{\operatorname{sech}\left(\frac{3\pi}{7}\right)}{3} + \frac{\operatorname{sech}\left(\frac{4\pi}{7}\right)}{4}\right.$$
$$- \frac{\operatorname{sech}\left(\frac{5\pi}{7}\right)}{5} - \frac{\operatorname{sech}\left(\frac{6\pi}{7}\right)}{6} + \frac{1}{8}\operatorname{sech}\frac{8\pi}{7} + \frac{\operatorname{sech}\left(\frac{9\pi}{7}\right)}{9}$$
$$- \frac{\operatorname{sech}\left(\frac{10\pi}{7}\right)}{10} + \frac{\operatorname{sech}\left(\frac{11\pi}{7}\right)}{11} - \frac{\operatorname{sech}\left(\frac{12\pi}{7}\right)}{12} - \frac{\operatorname{sech}\left(\frac{13\pi}{7}\right)}{13}$$
$$\left. + + - + - - \dots \dots\right\} = \frac{\pi}{4}.$$

got by putting $p = 7$ in (7.1).

* For similar results, see the conclusion of A.

NOTE ON § 1.

The following are particular cases of (1.4), (1,6) and (1.7).

If k is 0 or a positive integer, then

$$(1)\quad 1^{4k+3}\frac{\cosh\frac{\pi}{4}}{\cosh\frac{\pi}{2}}+3^{4k+3}\frac{\cosh\frac{3\pi}{4}}{\cosh\frac{3\pi}{2}}-5^{4k+3}\frac{\cosh\frac{5\pi}{4}}{\cosh\frac{5\pi}{2}}-7^{4k+3}\frac{\cosh\frac{7\pi}{4}}{\cosh\frac{7\pi}{2}}$$
$$+\ +\ -\ -\ \ldots\ \ldots = 0.$$

$$(2)\quad 1^{4k+1}\frac{\sinh\frac{\pi}{4}}{\cosh\frac{\pi}{2}}-3^{4k+1}\frac{\sinh\frac{3\pi}{4}}{\cosh\frac{3\pi}{2}}-5^{4k+1}\frac{\sinh\frac{5\pi}{4}}{\cosh\frac{5\pi}{2}}+7^{4k+1}\frac{\sinh\frac{7\pi}{4}}{\cosh\frac{7\pi}{2}}$$
$$+\ -\ -\ +\ \ldots\ \ldots = 0.$$

$$(3)\quad \frac{\cosh\frac{\pi}{4}}{\cosh\frac{\pi}{2}}+\frac{1}{3}\,\frac{\cosh\frac{3\pi}{4}}{\cosh\frac{3\pi}{2}}-\frac{1}{5}\,\frac{\cosh\frac{5\pi}{4}}{\cosh\frac{5\pi}{2}}-\frac{1}{7}\,\frac{\cosh\frac{7\pi}{4}}{\cosh\frac{7\pi}{2}}\ \ldots\ \ldots = \frac{\pi}{4\sqrt{2}}.$$

(1) and (2) are also easily obtained as particular cases of formula XXIV of A, while (3) is a consequence of the last formula in B.

In order to calculate the second member of (1.7) we note that the substitution of $y = \frac{1}{8}$ in the formula:

$$2\sum_{n=1}^{n=\infty}\frac{\sin(2n\pi y)}{(2n\pi)^{2m+1}}+\frac{(-1)^{m-1}}{\Gamma(2m+2)}\,\phi_{2m+1}(y)$$

$[0 < y < 1;\ m,$ *a* positive integer; $m \geqslant 1;\ \phi_n(x)$,

the Bernoullian polynomial of degree n],

leads to

$$\sqrt{2}\,\frac{L_2(2m+1)}{(2\pi)^{2m+1}}+\frac{2}{(2\pi)^{2m+1}}\quad\frac{\eta\,(1+2m)}{2^{3m+1}}=\frac{(-1)^{m-1}}{\Gamma(2m+2)}\,\phi_{2m+1}\left(\frac{1}{8}\right)$$

when $m \geqslant 1$.

Also $L_2(1) = \dfrac{\pi}{2\sqrt{2}}$, and thus the second member of (1.7) is of the form $A_k\,\pi^k$ where A_k is a rational number and k is a positive integer.

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NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

M. BHIMASENA RAO; G. A. SRINIVASAN, M.A., L.T.

AND

N. SUNDARAM IYER, M.A.

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Notes and Questions.

On the Nature of a Conic represented by the General Equation of the Second Degree.

In text-books on Co-ordinate Geometry, the general equation of the second degree is reduced to one or other of certain canonical forms, by a change of origin and axes. The shapes of the corresponding curves are then deduced from the special forms of the equations in each case. In this note, the procedure followed is just the opposite, that is, I deduce the condition for a parabola, ellipse (including the case of a circle), hyperbola, or a pair of straight lines, from the known shape of these curves, or rather from the nature of their intersections with a system of parallel lines.

The method depends on the following algebraical results which are proved in any text-book on Algebra :—

(1) If k_1, k_2 are the two roots (supposed to be real) of

$$Lk^2 + 2Mk + N = 0,$$

the quadratic $Lk^2 + 2Mk + N$ has the same sign as L for values of k outside the range (k_1, k_2) and a different sign for values of k in the range (k_1, k_2.)

(2) If $k_1 = k_2$ there is only one distinct root, and in this case $Lk^2 + 2Mk + N$ has the same sign as L for all values of k except k_1. This happens if $M^2 = LN$.

(3) Another case in which $Lk^2+2Mk+N=0$ has only one distinct real root k_1 is when $L=0$. In this case the quadratic changes sign as k passes through k_1.

Let us consider the intersections of

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

with the line $y = k$ for different values of k. The abscissae of the intersections are given by

$$ax^2 + 2x(hk + g) + bk^2 + 2fk + c = 0.$$

$$\therefore \quad x = -\frac{hk+g}{a} \pm \frac{1}{a}\left[(h^2 - ab)k^2 + 2(hg - af)k + (g^2 - ac)\right]^{\frac{1}{2}}$$

$$= -\frac{hk+g}{a} \pm \frac{1}{a}(Lk^2 + 2Mk + N)^{\frac{1}{2}}.$$

(*a*) If the equation represents a pair of real intersecting lines, both the abscissae are real and distinct for all values of k except one, in which exceptional case, the abscissae of intersection are equal. Hence $Lk^2 + 2Mk + N$ should vanish for only one value of k and should be positive for all other values. This is case (2) given above, and requires that $M^2 - LN = 0$ and L positive :

i.e. $\quad h^2 - ab > 0, \qquad \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$

unless a be zero. If a be zero we have only to consider the parallels $x = k$ instead of $y = k$.

(*b*) If the equation represents a pair of real parallel lines, the abscissae of the intersections should have a difference which is independent of k

This requires that $Lk^2 + 2Mk + N$ be independent of k.

$\therefore \;$ L = 0 and M = 0; $\quad$ *i.e.*, $h^2 - ab = 0$ and $hg - af = 0$.

These conditions may be shown to be equivalent to $\Delta = 0$ and $h^2 = ab$.

(*c*) If the curve be a parabola, the abscissae of intersections are coincident for one value of k, say k_1 and are real and distinct for values of k on one side of k_1, while there are no real intersections for values of k on the other side of k_1. This requires that $Lk^2 + 2Mk + N = 0$ for one value of k and that $Lk^2 + 2Mk + N$ should change sign as k passes through the value k_1. By (3) above, the condition for this is L = 0, *i.e*,

$$h^2 - ab = 0.$$

(*d*) If the curve be an ellipse, the abscissae of intersections are equal for two values of k; they should be real and distinct when k lies between k_1 and k_2; while there are no intersections for other values of k.

This means that $Lk^2 + 2Mk + N = 0$ has two real distinct roots, say k_1 and k_2. When k lies between k_1 and k_2 the expression should be positive and negative for other values. By (1) above, this will happen when L is negative, *i.e.* $h^2 - ab < 0$.

(*e*) We may similarly prove that if the curve is a hyperbola, $h^2 - ab > 0$.

(*f*) If the curve represents a circle, it is easily seen from a figure that the sum of the abscissae of the intersections is the same for every k.

This requires that $-(hk + g)/a$ be independent of k. Hence $h = 0$.

Again if the line $y = k$ touches the circle for the values k_1 and k_2 then $k_1 \sim k_2$ is equal to the diameter of the circle.

$$\therefore \qquad \frac{M^2 - NL}{L^2} = r^2,\ r \text{ being the radius of the circle,}$$

$$i.e., \qquad r^2(h^2 - ab)^2 = (hg - af)^2 - (h^2 - ab)(g^2 - ac).$$

Since $h = 0$

$$r^2a^2b^2 = a^2f^2 + ab(g^2 - ac).$$

Similarly by considering the parallels $x = l$, we may prove

$$r^2a^2b^2 = b^2g^2 + ab(f^2 - bc).$$

$$\therefore \qquad a^2f^2 + abg^2 - a^2bc = b^2g^2 + abf^2 - ab^2c$$

$$i.e., \qquad (a - b)(abc - af^2 - bg^2) = 0. \quad \therefore a = b,$$

since, if $abc - af^2 - bg^2$ were to vanish (in addition to $h = 0$), the condition for a pair of straight lines would be satisfied.

T. S. VENKATARAMAN.

Certain Criteria in regard to a triangle in Cartesian Geometry.

Consider a triangle ABC whose sides are given by the equations

$$y = m_r x + c_r \qquad (r = 1, 2, 3) \qquad \ldots \quad (1)$$

in rectangular Cartesian co-ordinates. Let the angle made by the line $y = mx$ with the x-axis be denoted by α, where $-\frac{\pi}{2} < \alpha \leqslant \frac{\pi}{2}$ and $\tan \alpha = m$. If $m_1 > m_2 > m_3$ so that $\alpha_1 > \alpha_2 > \alpha_3$ it is seen that the angles A, B, C of the triangle are $\alpha_2 - \alpha_3$, $\pi - (\alpha_1 - \alpha_3)$, $\alpha_1 - \alpha_2$. Generally, whatever be the order of the quantities m_1, m_2, m_3, we may state that the angle A is

$$|\alpha_2 - \alpha_3| \text{ or } \pi - |\alpha_2 - \alpha_3|$$

according as α_1 does not or does lie between α_2 and α_3. ... (2)

THEOREM 1. The angle A is acute, right or obtuse according as

$$(m_1 - m_2)(m_1 - m_3)(1 + m_2 m_3) \gtreqless 0. \qquad \ldots \quad (3)$$

For, from (2) we see that

$$\tan A = \frac{|m_2 - m_3|}{1 + m_2 m_3} \text{ or } \frac{-|m_2 - m_3|}{1 + m_2 m_3}$$

according as m_1 does not or does lie between m_2 and m_3.

Hence when

$$(m_1 - m_2)(m_1 - m_3) > 0,$$

angle A is acute, right or obtuse according as $1 + m_2 m_3 \gtreqless 0$

and when $$(m_1 - m_2)(m_1 - m_3) < 0$$

angle A is acute, right or obtuse according as $1 + m_2 m_3 \lesseqgtr 0$.

Hence the result (3).

Similarly angle B is acute, right or obtuse according as

$$(m_2 - m_3)(m_2 - m_1)(1 + m_3 m_1) \gtreqless 0 \qquad \ldots \quad (4)$$

and angle C is acute, right or obtuse according as

$$(m_3 - m_1)(m_3 - m_2)(1 + m_1 m_2) \gtreqless 0. \qquad \ldots \quad (5)$$

Combining results (3), (4) and (5), we get

THEOREM 2. The triangle ABC is acute-angled, right-angled or obtuse-angled according as*

$$(1 + m_1 m_2)(1 + m_2 m_3)(1 + m_3 m_1) \lesseqgtr 0.$$

THEOREM 3. The origin is in the interior angle A or the exterior angle A according as

$$c_2 c_3 (m_1 - m_2)(m_1 - m_3) \lessgtr 0$$

Take the triangle formed by

$$y = m_1 x, \; y = m_2 x + c_2 \text{ and } y = m_3 x + c_3.$$

The origin is in the interior angle A or exterior angle A according as the points of intersection of $y = m_1 x$ with the other two lines lie on opposite sides or on the same side of the origin. Hence the result.

Let L_r denote $(y - m_r x - c_r)/\sqrt{1 + m_r^2}$. Then for points in the interior angle A,

$$L_2 L_3 (m_1 - m_2)(m_1 - m_3) < 0.$$

i.e., L_2 and L_3 have the same sign if $(m_1 - m_2)(m_1 - m_3) < 0$ and have opposite signs if $(m_1 - m_2)(m_1 - m_3) > 0$.

Hence we have the following :

THEOREM 4. The internal bisector of the angle A is given by

$$L_2 = L_3 \text{ or } L_2 = -L_3$$

according as $(m_1 - m_2)(m_1 - m_3) \lessgtr 0$.

K. SATYANARAYANA.

* Vide *J.I.M.S.*, Vol. XVII, No. 3, p. *46* for another method of arriving at this result.

Solutions.

Question 1028.

(N. P. PANDYA) :—Two tangents to a parabola are fixed and a third one is variable. Find the locus of the centroid of a triangle inscribed in the parabola having its sides parallel to the three tangents.

Solution by N. G. Leather.

Let the parabola be $y^2 = 4\,ax$.

Let the ordinate of the vertex of the triangle, which is the meet of the sides parallel to the two fixed tangents, be k.

Let the ordinates of the points of contact of the two fixed tangents be y_1 and y_2. Then the ordinates of the other two vertices of the triangle will be $(2y_1 - k)$ and $(2y_2 - k)$

The co-ordinates of the centroid of the triangle will be given by

$$x = \frac{k^2 + (2y_1 - k)^2 + (2y_2 - k)^2}{12a},\quad y = \frac{k + (2y_1 - k) + (2y_2 - k)}{3}$$

$$\therefore \qquad 3k^2 - 4k\,(y_1 + y_2) + 4\,(y_1^2 + y_2^2 - 3\,ax) = 0$$

and

$$k = 2\,(y_1 + y_2) - 3y.$$

Eliminating k, and simplifying, we get for the locus

$$\left[y - \tfrac{4}{9}(y_1 + y_2)\right]^2 = \tfrac{4a}{9}\left[x - \frac{2}{9a}(y_1^2 - y_1 y_2 + y_2^2)\right]$$

Question 1042.

(MARTYN M. THOMAS, M.A.) :—The envelope of the curves

$$y \sin^2\theta = a \sin\theta - x \sin\theta \cos\theta - (x - a\cos\theta)^2 / c$$

where θ is variable, consists of two parts. Give a dynamical interpretation of the result.

Partial Solution by B. Nagendranath Dutt.

Differentiating

$$y \sin^2 \Theta = a \sin \Theta - x \sin \Theta \cos \Theta - (x - a \cos \Theta)^2 / c \quad \ldots \quad \text{(A)}$$

with respect to Θ, we have

$$2y \sin \Theta \cos \Theta = a \cos \Theta - x (\cos^2 \Theta - \sin^2 \Theta) - \frac{2a \sin \Theta}{c} (x - a \cos \Theta) \quad \ldots \quad \text{(B)}$$

Eliminating y between (A) and (B), we have, after simplifying

$$\left(\frac{2 \cos \Theta}{c} x + \sin \Theta - \frac{2a}{c}\right) \left(x - a \cos \Theta\right) = 0.$$

Case I. $x = a \cos \Theta$ with (A) gives $y = a \sin \Theta$,

or, for the envelope, $x^2 + y^2 = a^2$.

Case II. $x = \dfrac{-c \sin \Theta + 2a}{2 \cos \Theta}$, when substituted in (A) gives for

y after simplification the value

$$y = \frac{a^2}{c} + \frac{c}{4} - \frac{1}{4c \cos^2 \Theta} (2a - c \sin \Theta)^2$$

or

$$y = \frac{a^2}{c} + \frac{c}{4} - \frac{x^2}{c} \quad \ldots \quad \ldots \quad \text{(C)}$$

which is the corresponding envelope.

Remarks by G. A. Srinivasan.

Suppose a circular disc of radius a with its plane vertical rotates about its centre O with uniform angular velocity ω given by $\omega^2 = cg / 2a^2$. Consider the trajectories of particles thrown off from its rim. A particle leaving the disc from any point P on its rim starts with a velocity $a\omega$ along the tangent at P. The parabola described by this particle is seen to be given by the equation (A), where the horizontal and vertical lines through O are the axes of co-ordinates and Θ is the angle made by OP with the x-axis. One envelope of these trajectories is obviously the disc itself. The other envelope is the parabola (C) above.

Question 1048.

(V. THIRUVENKATACHARYA) :—Establish the following asymptotic relations :—

$$\int_0^1 \frac{\sin\left(\log \frac{1}{y^p}\right)}{(1-xy)\log\left(\frac{1}{y}\right)}\,dy = \tan^{-1}\left(\frac{p}{1}\right) + x\tan^{-1}\left(\frac{p}{2}\right) + x^2\tan^{-1}\left(\frac{p}{3}\right) + \dots\dots$$

$$\int_0^1 \frac{\cos\left(2\log\frac{1}{y}\right)}{(1-xy)\sqrt{\log\frac{1}{y}}}\,dy = \sqrt{\pi}\left\{\frac{e^{-1}}{\sqrt{1}} + x\frac{e^{-\frac{1}{2}}}{\sqrt{2}} + x^2\frac{e^{-\frac{1}{3}}}{\sqrt{3}} + x^3\frac{e^{-\frac{1}{4}}}{\sqrt{4}} + \dots\right\}$$

when $|x| < k \leqslant 1$.

Solution by B. Nagendranath Dutt

In the first integral put

$$\log\frac{1}{y} = z;\; y = e^{-z} \text{ and } dy = -e^{-z}\,dz.$$

Then it becomes

$$\int_0^\infty \frac{\sin pz\; e^{-z}}{z(1-x e^{-z})}\,dz = \int_0^\infty \frac{\sin pz}{z} e^{-z}(1 + x e^{-z} + x^2 e^{-2z} + \dots)\,dz.$$

if $|x| < k \leqslant 1$. In the general term of the above integration the co-efficient of x^r is

$$\int_0^\infty \frac{e^{-(r+1)z}\sin pz}{z}\,dz = \tan^{-1}\left(\frac{p}{r+1}\right).^*$$

Hence the given integral $= \sum_0^\infty x^r \tan^{-1}\dfrac{p}{r+1}$.

In the second part the integral is incorrect : it must be

$$\int_0^1 \frac{\cos(2\sqrt{\log 1/y})}{(1-xy)\sqrt{\log 1/y}}\,dy \quad \dots \quad \dots \quad \text{(A)}$$

* *Vide* Williamson : *Integral Calculus*, p. 146, Ex. 2.

To show that this is equal to the series on the right, put

$$\sqrt{\log\frac{1}{y}} = z;\ i.e.,\ y = e^{-z^2},\ dy = -2\,z\,e^{-z^2}\,dz.$$

Then the integral (A) becomes

$$2\int_0^\infty \frac{\cos 2z\, e^{-z^2}\, dz}{(1 - x\,e^{-z^2})}.$$

$$= 2\int_0^\infty \cos 2z\ e^{-z^2}(1 + x e^{-z^2} + x^2 e^{-2z^2} + \ldots)\,dz.$$

since $|\,x\,| < k \leqslant 1$.

The co-efficient of x^r in this series is

$$2\int_0^\infty \cos 2z\ e^{-z^2}\ e^{-rz^2}\,dz = 2\int_0^\infty e^{-(r+1)z^2}\cos 2z\,dz$$

$$= \frac{\sqrt{\pi}\, e^{-\frac{1}{r+1}}}{\sqrt{r+1}}\text{*}.$$

Hence the integral (A) is equal to

$$\sqrt{\pi}\ \sum_0^\infty \frac{e^{-\frac{1}{r+1}}}{\sqrt{(r+1)}}\,x^r.$$

Now let us proceed to evaluate the integral

$$\int_0^1 \frac{\cos(2\log 1/y)}{(1 - xy)\sqrt{\log 1/y}}\,dy \qquad \ldots \qquad \ldots \quad \text{(B)}$$

Making the substitution $y = e^{-z}$, we find the co-efficient of x^r in the series for this is

$$\int_0^\infty e^{-(r+1)z}\cos 2z\,.\,z^{-\frac{1}{2}}\ dz = \frac{\Gamma(\frac{1}{2})}{\{(r+1)^2 + 4\}^{\frac{1}{4}}}\cos\frac{\theta_r}{2},$$

where $$\theta_r = \tan^{-1}\frac{2}{r+1}.\ \dagger$$

Hence the integral (B) is equal to

$$\sqrt{\pi}\ \sum_0^\infty \frac{\cos\frac{\theta_r}{2}}{\sqrt[4]{\{r^2 + 2r + 5\}}}\,.\,x^r;\quad \left(\theta_r = \tan^{-1}\frac{2}{r+1}\right).$$

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* *Vide* Williamson, p. 152.
† *Vide* Williamson, p. 166.

Question 1255.

(T. VIJAYARAGHAVAN) :—Show that

$$\lim_{m \to \infty} B\left[m, \frac{\log\log m - \log\log\log m - k}{\log m}\right] = e^k$$

where B (p, q) denotes as usual the integral

$$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx.$$

Solution by S. D. Chowla.

Let $$n = \frac{\log\log m - \log\log\log m - k}{\log m}$$

then $n \to 0$ as $m \to \infty$ and hence

$$\lim_{m \to \infty} n\,\Gamma(n) = 1. \qquad \ldots \qquad \ldots \qquad (1)$$

Now the required limit is

$$\lim_{m \to \infty} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Using (1) and Stirling's formula, *viz.*, $\Gamma(x) \sim e^{-x} x^x (2\pi/x)^{\frac{1}{2}}$, we have

$$B(m, n) \sim \frac{e^{-m}.m^m.(2\pi/m)^{\frac{1}{2}}.1/n}{e^{-m-n}(m+n)^{m+n}(2\pi/m+n)^{\frac{1}{2}}}.$$

$$\sim \frac{e^n}{n}.\frac{1}{m^n}.\frac{1}{(1+n/m)^{m+n}}$$

$$\sim \frac{e^n}{n}.\frac{1}{m^n}.\frac{1}{e^n} \sim \frac{1}{nm^n}.$$

Now from the definition of n, we get

$$\frac{1}{nm^n} = e^k.\frac{\log\log m}{\log\log m - \log\log\log m - k}$$

$$\sim e^k.$$

Hence the result.

Question 1280.

(M. BHEEMASENA RAO and the late C. KRISHNAMACHARI) :—

If $S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \ldots$ and γ is Euler's constant, show that

$$\int_0^\infty e^{-x} (\log x)^3 \, dx = -(\gamma^3 + 3\gamma S_2 + 2 S_3)$$

$$\int_0^\infty e^{-x} (\log x)^4 \, dx = (\gamma^4 + 6\gamma^2 S_2 + 8\gamma S_3 + 3 S_2^2 + 6 S_4)$$

and show how to evaluate generally

$$\int_0^\infty e^{-x} (\log x)^n \, dx.$$

Remarks by S. D. Chowla.

The cases $n = 3, 4$ have been worked out by Mr. M. M. Thomas in this Journal (*vide* Vol. XV, p. *178*). A remarkably quick way of getting

$$\int_0^\infty e^{-x} (\log x)^n \, dx$$

has recently been given by Appell (*Acta Mathematica* . **45** . 3, 4, pages 287—302). Writing

$$\Gamma^n (1) = \int_0^\infty e^{-x} (\log x)^n \, dx = P_n (\gamma),$$

Appell proves that $P_n(\gamma)$ is a polynomial in γ, such that

$$P_{n+1}(\gamma) = (-1).(1+n).\int^\gamma P_n(\gamma)\, d\gamma + \text{a term independent of } \gamma$$

and a formula for the term independent of γ has also been given, *e.g.* from the cases $n = 3$, $n = 4$, we get *at once*

$$\int_0^\infty e^{-x} (\log x)^5 \, dx = -\left[\gamma^5 + 10\gamma^3 S_2 + 20\gamma^2 S_3 + 5\gamma\,(3 S_2^2 + 6 S_4) + 20 S_2 S_3 + 24 S_5\right].$$

Question 1303.

(V. THIRUVENKATACHARYA) :—Show that

$$\int_0^\infty \left(\frac{e^{-t}}{t\,(e^t+1)} - \frac{e^{-t}}{e^{2t}-1}\right) dt = \frac{1}{2}\log 4\pi + 1 + \frac{\gamma}{2}.$$

where γ is Eulei's constant.

Solution by S. D. Chowla and S. Mahadevan.

The integral is equal to

$$\int_0^\infty \left[\frac{e^{-t}}{t\,(e^t+1)} - \frac{e^{-t}}{2t}\right] dt + \int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{2t}\right) dt$$

$$+ \int_0^\infty \left(\frac{e^{-2t}}{2t} - \frac{e^{-t}}{e^{2t}-1}\right) dt = I_1 + I_2 + I_3.$$

Each of these is a convergent integral.

Now

$$I_1 = \int_0^\infty \frac{e^{-}\,(1-e^t)}{2t\,(e^t+1)}\, dt = -\int_0^\infty \frac{e^{-t}}{2t}\tanh\left(\frac{t}{2}\right) dt$$

$$= -\int_0^\infty \frac{e^{-2t}}{2t}\tanh t\, dt = -\tfrac{1}{2}\{-\log 2 + \log \pi\}$$

since

$$\int_0^\infty \frac{e^{-ax}}{x}\tanh x\, dx = \log\frac{a}{4} + 2\log\left\{\frac{\Gamma\left(\frac{a}{4}\right)}{\Gamma\left(\frac{a}{4}+\frac{1}{2}\right)}\right\}$$

(Bromwich : *Theory of Infinite Series*, 1st Edition, p. 478, Ex. 52).

Also $$I_2 = \tfrac{1}{2}\log 2$$

Lastly $$I_3 = \tfrac{1}{2}\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\frac{1}{2}t}}{e^t-1}\right) dt = 1 - \tfrac{1}{2}\gamma - \log 2$$

since $$\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tx}}{e^t-1}\right) dt = \frac{\Gamma'(1+x)}{\Gamma(1+x)}$$

(Bromwich : *Theory of Infinite Series*, p. 466).

Hence the integral of Q. 1303 is equal to

$$I_1 + I_2 + I_3 = 1 - \tfrac{1}{2}\gamma - \tfrac{1}{2}\log\pi.$$

The value given in the question is thus incorrect.

Question 1391.

(V. THIRUVENKATACHARYA):—Show that

$$\text{(i)}\quad \sum_{n=0}^{\infty} \frac{\cos(6n+4)\dfrac{\theta}{3}}{\dfrac{6n+4}{3}} = \tfrac{1}{4}\log \frac{\sin\theta}{4\sin^3\dfrac{\theta}{3}} - \frac{\pi}{4\sqrt{3}}.$$

$$\text{(ii)}\quad \sum_{n=0}^{\infty} \frac{\sin(6n+4)\dfrac{\theta}{3}}{\dfrac{6n+4}{3}} = \tfrac{3}{4}\log \frac{2\left(\sqrt{3}-\tan\dfrac{\theta}{3}\right)}{2\left(\sqrt{3}+\tan\dfrac{\theta}{3}\right)} + \frac{\pi}{4} - \frac{\sqrt{3}}{4}\log 2.$$

where $0 < \theta < \pi$.

Solution by M. V. Seshadri, S. D. Chowla, S. Mahadevan and Kanwar Bahadur.

The right-hand side of (ii) must be

$$\frac{\sqrt{3}}{4}\log\left\{\frac{\sqrt{3}-\tan\dfrac{\theta}{3}}{\sqrt{3}+\tan\dfrac{\theta}{3}}\right\} + \frac{\pi}{4}.$$

Now

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\ \dots$$

$$-\omega\log(1-\omega x) = \omega^2 x + \frac{\omega^3 x^2}{2} + \frac{\omega^4 x^3}{3} + \dots\ \dots$$

and

$$-\omega^2\log(1-\omega^2 x) = \omega^4 x + \frac{\omega^6 x^2}{2} + \frac{\omega^8 x^3}{3} + \dots\ \dots$$

where ω, ω^2 are the imaginary cube-roots of unity. Adding these equations and noting that $1 + \omega^r + \omega^{2r} = 0$ or 3 according as r is not or is a multiple of 3, we get

$$3\left[\frac{x^2}{2} + \frac{x^5}{5} + \dots\ \dots + \frac{x^{3n+2}}{3n+2} + \dots\ \dots\right]$$

$$= -\left[\log(1-x) + \omega\log(1-\omega x) + \omega^2\log(1-\omega^2 x)\right] \dots\ \dots (1).$$

In this equation write $x = e^{2i\phi}$; then if $0 < \phi < \pi/3$, none of the logarithms on the right becomes infinite. The left-hand side becomes $3(C + iS)$ where

$$C \equiv \sum_{0}^{\infty} \frac{\cos(3n+2)2\phi}{3n+2}, \quad S \equiv \sum_{0}^{\infty} \frac{\sin(3n+2)2\phi}{3n+2}.$$

To separate the right-hand side of (1) into real and imaginary parts, we note that if $0 < \phi < \pi$,

$$-\log(1 - e^{2i\phi}) = -\log(2\sin\phi) + i\left[\frac{\pi}{2} - \phi\right].$$

Hence, since $0 < \phi < \frac{\pi}{3}$, we have also,

$$-\log(1 - \omega e^{2i\phi}) = -\log\left\{2\sin\left[\phi + \frac{\pi}{3}\right]\right\} + i\left\{\frac{\pi}{2} - \phi - \frac{\pi}{3}\right\}$$

and

$$-\log(1 - \omega^2 e^{2i\phi}) = -\log\left\{2\sin\left[\phi + \frac{2\pi}{3}\right]\right\} + i\left\{\frac{\pi}{2} - \phi - \frac{2\pi}{3}\right\}$$

Substituting these on the right-hand side of (1), we get

$$\begin{aligned} 3C = &-\log(2\sin\phi) \\ &+ \tfrac{1}{2}\log\left\{2\sin\left[\phi + \frac{\pi}{3}\right]\right\} + \tfrac{1}{2}\log\left\{2\sin\left[\phi + \frac{2\pi}{3}\right]\right\} \\ &- \frac{\sqrt{3}}{2}\left[\frac{\pi}{2} - \phi - \frac{\pi}{3}\right] + \frac{\sqrt{3}}{2}\left[\frac{\pi}{2} - \phi - \frac{2\pi}{3}\right] \\ &= \tfrac{1}{2}\log\left(\frac{\sin 3\phi}{4\sin^3\phi}\right) - \frac{\pi}{2\sqrt{3}}, \end{aligned}$$

on simplification.

Similarly,

$$\begin{aligned} 3S = &\frac{\sqrt{3}}{2}\log\left\{2\sin\left(\phi + \frac{2\pi}{3}\right)\right\} - \frac{\sqrt{3}}{2}\log\left\{2\sin\left(\phi + \frac{\pi}{3}\right)\right\} \\ &+ \left(\frac{\pi}{2} - \phi - \frac{\pi}{4} + \frac{\phi}{2} + \frac{\pi}{6} - \frac{\pi}{4} + \frac{\phi}{2} + \frac{\pi}{3}\right) \\ = &\frac{\sqrt{3}}{2}\log\left(\frac{\sqrt{3} - \tan\phi}{\sqrt{3} + \tan\phi}\right) + \frac{\pi}{2}. \end{aligned}$$

The series (i) and (ii) are $\frac{3C}{2}$ and $\frac{3S}{2}$, with $\frac{\theta}{3}$ written for ϕ.

Question 1401.

(S. D. Chowla) :—With the usual notation* for Bernoulli's numbers, prove that

(1) $2\{{}^{2p}C_2B_2 - {}^{2p}C_4B_4 + {}^{2p}C_6B_6 - \dots + (-1)^p\,{}^{2p}C_{2p-2}B_{2p-2}\}$

is always an integer.

(2) $$\frac{B_2}{1.2} - {}^{2p}C_2\frac{B_4}{3.4} + {}^{2p}C_4\frac{B_6}{5.6} - \dots$$

$$+ (-1)^{p-1}\,{}^{2p}C_{2p-2}\frac{B_{2p}}{(2p-1)(2p)} = \frac{1}{2p+2} - \frac{1}{2(2p+1)}.$$

Solution by B. Achyutarama Sastri and K. V. Vedantam.

Consider

$$S_{2p} \equiv \frac{B_2}{2!}\cdot\frac{1}{(2p-2)!} - \frac{B_4}{4!}\,\frac{1}{(2p-4)!} + \dots + (-1)^{p-2}\frac{B_{2p-2}}{(2p-2)!}\cdot\frac{1}{2!}.$$

It is easily seen that S_{2p} is the co-efficient of x^{2p} in the series obtained by multiplying the right-hand sides of the two expansions

$$\frac{x}{e^x-1} - 1 + \frac{x}{2} = \frac{B_2}{2!}x^2 - \frac{B_4}{4!}x^4 + \dots + (-1)^{p-2}\frac{B_{2p-2}}{(2p-2)!}x^{2p-2} + \dots$$

and $$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots\dots$$

Hence S_{2p} is the co-efficient of x^{2p} in

$$(e^x-1)\left(\frac{x}{e^x-1} - 1 + \frac{x}{2}\right)$$

i.e., in $$1 + \frac{x}{2} + \left(\frac{x}{2} - 1\right)e^x.$$

Hence $$S_{2p} = \frac{1}{2(2p-1)!} - \frac{1}{(2p)!} = \frac{p-1}{(2p)!}.$$

Now the expression (1) in the question is $2\,(2p)!\,S_{2p}$ which is therefore equal to $2\,(p-1)$.

Expression (2) is $(2p)!\,S_{2p+2}$ which is equal to

$$\frac{p}{(2p+2)(2p+1)} \quad i.e., \quad \frac{1}{2p+2} - \frac{1}{2(2p+1)}.$$

---

* $B_2, B_4, B_6 \dots$ are those usually denoted by $B_1, B_2, B_3 \dots\dots$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1528. (T. VIJAYARAGHAVAN):—Suppose that a's and b's are positive numbers and that $\Sigma\, b_n$ is convergent. Prove that if

$$t_n = \sqrt{a_1 + \sqrt{a_2 + \dots + \sqrt{a_n}}}$$

then the set (t_n) is convergent whenever the series

$$\Sigma\, e^{-b_n 2^n}\, a_n\, (a_1 a_2 \dots a_n)^{-\frac{1}{2}}$$

is covergent. Also, show that this is a 'best possible' result.

[The above result is a more precise one than that given by Pólya and Szego; Problem No 163, I Abschn., *Aufgaben und Lehrsatze aus der Analysis 1*].

1529. (S. SIVASANKARANARAYANA PILLAI):—If u and v do not contain any prime to the same highest power, then

$$\phi\,(uv) = \Sigma\, \delta\, \phi\left(\frac{u}{\delta}\right)\, \phi\left(\frac{v}{\delta}\right)$$

where δ runs through all the common divisors of u and v and $\phi\,(x)$ denotes the number of numbers not exceeding x and prime to x.

1530. (R. VAIDYANATHASWAMY):—The function ϕ' (N) is defined to be zero when N is not a square, and to be equal to $\phi(\sqrt{N})$ when N is a square (ϕ being Euler's function). F (N) is equal to the least divisor δ of N, which is divisible by the complementary divisor N/δ, Shew that the following equality holds whenever no common prime factor of M, N, occurs in them with equal indices :

$$\phi'(MN) = \Sigma\, \phi'\left(\frac{M}{\delta}\right)\, \phi'\left(\frac{N}{\delta}\right)\, F(\delta),$$

the summation extending to the common divisors δ of M, N.

1531. (R. VAIDYANATHASWAMY):—If σ_t (N) is the sum of the tth powers of the divisors of N, shew that $\Sigma\, \sigma_t(\delta_1)\, \sigma_t(\delta_2)$ summed for all pairs of numbers δ_1, δ_2 which have the least common multiple N, is equal to $\sum \sigma_t\left(\frac{N^2}{\delta^2}\right) \delta^t$, summed for all divisors δ of N.

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of October and November 1928.

1 American Journal of Mathematics, **50**, 4 (2 copies).

2 American Mathematical Monthly, **35**, 7.

3 Annales de L'Ecole Normale Superieure, June and July, 1928.

4 Astrophysical Journal, **68**, 2.

5 Bulletin des Sciences Mathematiques, August to October, 1928.

6 Deutschen Mathematiker Vereinigung, Jahresbericht, **37**, 5 to 10.

7 Japanese Journal of Mathematics.

8 Journal of Mathematics, 1928, Vol. I.

9 Mathematical Gazette, 196, 197 (2 copies each).

10 Messenger of Mathematics, **57**, 11 (2 copies).

11 Monthly Notices of the Royal Astronomical Society, **88**, 9.

12 Philosophical Magazine and Journal of Science, **6**, 37.

13 Popular Astronomy, **36**, 8.

14 Proceedings of the Cambridge Philosophical Society, **24**, 4.

15 Proceedings of the London Mathematical Society, **28**, 5, 6 (2 copies).

16 Proceedings of the Royal Society of London, **120**, 786 (2 copies); **121**, 787.

17 Rendiconti del Circolo Matematico di Palermo, **52**, 2.

18 Transactions of the Royal Society of South Africa, **17**, 1.

Books.

1 Astrographic Catalogue 1900-0, Hyderabad Section.

2 A Cremona Group Isomorphism with the Group of the 27 lines on a Cubic Surface, by H. L. Black.

3 A certain General Type of Neumann-expansion and expansions of Confluent Hypergeometric Functions, by R. F. Graesser.

4 Calendar for 1928-29, University College of Southampton

5 Mathieu Functions, by S. C. Dhar

6 India through the Ages, University of Madras.

7 Political Theory of the Government of India, University of Madras.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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